Meson twist-4 parton distributions in terms of twist-2 distribution amplitudes at large N_c

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Abstract

We show that in the large N_c limit four-quark twist-4 distributions in the pion can be expressed in terms of twist-2 pion distribution amplitude. This allows us to compute the isospin-2 structure function of the pion $F_2^{I=2}(x_B)$ in the large N_c limit. The method can be easily applied to other mesons as well.

1. Matrix elements of four-quark operators appear *e.g.* in twist-4 contributions to the structure functions of deep–inelastic scattering [1, 2], or in the description of non-leptonic weak decays [3]. In this paper we consider the specific example of isospin-2 pion structure functions $F_2(x_B)$

$$F_2^{I=2} = F_2^{\pi^+} + F_2^{\pi^-} - 2F_2^{\pi^0} \,. \tag{1}$$

In this particular isospin combination of structure functions the contribution of twist-2 operators cancels out exactly and twist-4 contributions are dominant. It is remarkable that in the large N_c limit the twist-4 distribution function of quarks in the pion can be reduced to a convolution of twist-2 pion distribution amplitudes.

2. One can read the contribution of four-quark twist-4 operators to F_2 from [4, 5] (correcting obvious misprints)

$$\frac{1}{x_B} F_2^{twist-4}(x_B) = \frac{1}{Q^2} \int dx_1 dx dx_2 \, x_B$$

$$\times \left\{ [\Delta(x_2, x_1, x) + \Delta(y_2, y_1, x)] \, T_{qq}^{(+)}(x_1, x, x_2) \right.$$

$$\left. - [\Delta(x_2, y_1, x) + \Delta(y_2, x_1, x)] \, T_{qq}^{(-)}(x_1, x, x_2) \right\} \,. \tag{2}$$

Here $y_1 \equiv x - x_1$, $y_2 \equiv x - x_2$. We use the function Δ from ref. [4]

$$\Delta(x_1, x_2, x_3) = -\frac{1}{\pi} \text{Im} \left[\frac{1}{x_1 - x_B + i0} \frac{1}{x_2 - x_B + i0} \frac{1}{x_3 - x_B + i0} \right] \,. \tag{3}$$

The functions $T_{qq}^{(\pm)}(x_1, x, x_2)$ are defined as follows

$$T_{qq}^{(\pm)}(x_1, x, x_2) = T_{qq}(x_1, x, x_2) \pm \tilde{T}_{qq}(x_1, x, x_2) \tag{4}$$

with $T_{qq}, \tilde{T}_{qq}$ given by

$$\left.\begin{array}{l} T_{qq}(x_1, x, x_2) \\ \tilde{T}_{qq}(x_1, x, x_2) \end{array}\right\} = g^2 \frac{p^+}{4} \int \frac{dz_1^-}{2\pi} \frac{dz_2^-}{2\pi} \frac{dz^-}{2\pi}$$

$$\times \exp\left\{ip^+ \left[x_1 z_1^- + (x - x_1)z^- - (x - x_2)z_2^-\right]\right\}$$

$$\times \langle p|T\left\{\left[\bar{\psi}(0)\Gamma t^B \psi(z_2^-)\right]\left[\bar{\psi}(z^-)\Gamma t^B \psi(z_1^-)\right]\right\}|p\rangle. \tag{5}$$

Here g is the QCD coupling constant, $t^B = \frac{1}{2}\lambda^B$ are generators of the color group and

$$\Gamma = \left\{\begin{array}{ll} \gamma^+ Q_e & \text{for } T_{qq} \\ \gamma_5 \gamma^+ Q_e & \text{for } \tilde{T}_{qq} \end{array}\right., \tag{6}$$

where Q_e is the quark electric charge matrix. The light-cone components of a vector V^μ are defined as $V^\pm = \frac{1}{\sqrt{2}}(V^0 \pm V^3)$.

3. Generically the meson matrix elements of four-quark twist-4 operators are not related to matrix elements of two-quark operators. Here we show that in the limit of large number of colors N_c the matrix elements of four-quark operators can be expressed in terms of marix elements of two-quark operators. This observation allows us to express twist-4 contribution to $F_2(x)$ of the pion in terms of twist-2 distribution amplitudes.

A simple analysis of leading in N_c Feynman diagrams [6], corresponding to the meson matrix element

$$\langle M_1|(\bar{\psi}O_1\psi)(\bar{\psi}O_2\psi)|M_2\rangle \tag{7}$$

with matrices O_i being unity in the color subspace, shows that this matrix elements can be factorized in the limit of large number of colors as

$$\langle M_1|(\bar{\psi}O_1\psi)(\bar{\psi}O_2\psi)|M_2\rangle \stackrel{N_c\to\infty}{\longrightarrow} \langle M_1|M_2\rangle\langle 0|(\bar{\psi}O_1\psi)(\bar{\psi}O_2\psi)|0\rangle$$

$$+ \langle M_1|(\bar{\psi}O_1\psi)|0\rangle \langle 0|(\bar{\psi}O_2\psi)|M_2\rangle + \langle M_1|(\bar{\psi}O_2\psi)|0\rangle \langle 0|(\bar{\psi}O_1\psi)|M_2\rangle. \tag{8}$$

Let us define the connected part of the matrix element as follows

$$\langle M_1|(\bar{\psi}O_1\psi)(\bar{\psi}O_2\psi)|M_2\rangle_{connected}$$

$$\equiv \langle M_1|(\bar{\psi}O_1\psi)(\bar{\psi}O_2\psi)|M_2\rangle - \langle M_1|M_2\rangle\langle 0|(\bar{\psi}O_1\psi)(\bar{\psi}O_2\psi)|0\rangle. \tag{9}$$

If mesons M_1 and M_2 have different momenta or different internal quantum numbers then $\langle M_1|M_2\rangle = 0$ and there is no difference between the matrix elements and its connected part. But if $|M_1\rangle = |M_2\rangle$ then the nonconnected matrix element $\langle M_1|(\bar{\psi}O_1\psi)(\bar{\psi}O_2\psi)|M_2\rangle$ is divergent and we must consider the connected part. Now the factorization (8) takes the form

$$\langle M_1|(\bar{\psi}O_1\psi)(\bar{\psi}O_2\psi)|M_2\rangle_{connected}$$

$$\stackrel{N_c\to\infty}{\longrightarrow} \langle M_1|(\bar{\psi}O_1\psi)|0\rangle \langle 0|(\bar{\psi}O_2\psi)|M_2\rangle + \langle M_1|(\bar{\psi}O_2\psi)|0\rangle \langle 0|(\bar{\psi}O_1\psi)|M_2\rangle. \tag{10}$$

We want to investigate the large N_c behavior of the pion distribution functions. They are defined in terms of matrix elements

$$\langle M_1|T\left\{\left[\bar{\psi}(z_1')\Gamma_1 t^B \psi(z_1)\right]\left[\bar{\psi}(z_2')\Gamma_2 t^B \psi(z_2)\right]\right\}|M_2\rangle. \tag{11}$$

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We remind that

$$\mathrm{Sp}(t^A t^B) = \frac{1}{2}\delta^{AB}\,, \tag{12}$$

$$\sum_B t^B_{ij} t^B_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right)\,. \tag{13}$$

In the large N_c limit this reduces to

$$\sum_B t^B_{ij} t^B_{kl} \to \frac{1}{2}\delta_{il}\delta_{jk}\,. \tag{14}$$

Hence for any color singlet matrices $\Gamma_{1,2}$ we have

$$\langle M_1|T\left\{\left[\bar{\psi}(z_1')\Gamma_1 t^B \psi(z_1)\right]\left[\bar{\psi}(z_2')\Gamma_2 t^B \psi(z_2)\right]\right\}|M_2\rangle$$

$$\xrightarrow{N_c \to \infty} -\frac{1}{2}\langle M_1|T\left\{\mathrm{Sp}\left[\psi(z_2)\otimes\bar{\psi}(z_1')\right]\Gamma_1\left[\psi(z_1)\otimes\bar{\psi}(z_2')\right]\Gamma_2\right\}|M_2\rangle\,. \tag{15}$$

We stress that here in $\bar{\psi}(z_1')\otimes\psi(z_2)$ the tensor product refers to the spin flavor indices while the color indices are contracted. The trace Sp also refers only to spin-flavor indices.

Applying the large N_c factorization formula (10) to the matrix element in the rhs of (15) one obtains

$$\langle M_1|T\left\{\left[\bar{\psi}(z_1')\Gamma_1 t^B \psi(z_1)\right]\left[\bar{\psi}(z_2')\Gamma_2 t^B \psi(z_2)\right]\right\}|M_2\rangle_{connected}$$

$$\xrightarrow{N_c \to \infty} -\frac{1}{2}\mathrm{Sp}\left\{\langle M_1|T\left[\psi(z_2)\otimes\bar{\psi}(z_1')\right]|0\rangle\,\Gamma_1\,\langle 0|T\left[\psi(z_1)\otimes\bar{\psi}(z_2')\right]|M_2\rangle\,\Gamma_2\right\}$$

$$-\frac{1}{2}\mathrm{Sp}\left\{\langle 0|T\left[\psi(z_2)\otimes\bar{\psi}(z_1')\right]|M_2\rangle\,\Gamma_1\,\langle M_1|T\left[\psi(z_1)\otimes\bar{\psi}(z_2')\right]|0\rangle\,\Gamma_2\right\}\,. \tag{16}$$

Now we apply the general large N_c eq. (16) to matrix elements entering in definitions of twist-4 parton distributions (5). We obtain the following result for these functions

$$T^{(+)}_{qq}(x_1, x, x_2) \equiv \left[T_{qq}(x_1, x, x_2) + \tilde{T}_{qq}(x_1, x, x_2)\right]$$

$$= -g^2 \frac{F_\pi^2}{8}\left[\delta(x_1 - x_2 - 1)\phi_\pi(x_1 - x)\phi_\pi(x_1) + \delta(x_2 - x_1 - 1)\phi_\pi(x_2 - x)\phi_\pi(x_2)\right]$$

$$\times\left(-\frac{4}{9}\delta^{ab} + \delta^{a3}\delta^{b3}\right)\,. \tag{17}$$

Here $\phi_\pi(x)$ is the pion *twist-2* distribution amplitude defined as

$$\frac{1}{F_\pi}\int\frac{d\lambda}{2\pi}e^{ix\lambda(P\cdot n)}\langle 0|\bar{\psi}(\lambda n)\hat{n}\gamma_5 \frac{\tau^b}{2}\psi(0)|\pi^a(P)\rangle = i\delta^{ab}\phi_\pi(x)\,. \tag{18}$$

Here $F_\pi \approx 93$ MeV is the pion decay constant and a, b are isospin indices. Also it is easy to see from definition eqs. (4,5) that at leading order of the $1/N_c$ expansion

$$T^{(-)}_{qq}(x_1, x, x_2) = 0\,. \tag{19}$$

We see that in the large N_c limit the meson twist-4 four-quark distributions are expressible in terms of twist-2 meson distribution amplitudes. Now we are in a position to compute the twist-4 part of $F_2(x_B)$. We insert expressions (17,19) into eq. (2)

$$F_2^{twist-4}(x_B) = -g^2 \frac{x_B^2}{Q^2} \frac{F_\pi^2}{4} \left(-\frac{4}{9}\delta^{ab} + \delta^{a3}\delta^{b3}\right)$$

$$\times \left\{ 2\phi_\pi(x_B) \int dx_1 \phi_\pi(x_1) \frac{1}{x_1} \right.$$

$$\left. - \int dx_1 dy_1 \phi_\pi(y_1)\phi_\pi(x_1)\delta(x_1 - y_1 - x_B) \left[\frac{1}{(1-y_1)y_1} + \frac{1}{(1-x_1)x_1}\right] \right\}. \tag{20}$$

From this equation we can easily compute pion isospin-2 structure function

$$F_2^{I=2} = \left[F_2^{twist-4}\right]^{\pi^+} + \left[F_2^{twist-4}\right]^{\pi^-} - 2\left[F_2^{twist-4}\right]^{\pi^0}. \tag{21}$$

Here we have taken into account that the twist-2 contributions are cancelled in the rhs of eq. (1). The isospin factor of eq. (17) yelds

$$\left(-\frac{4}{9}\delta^{ab} + \delta^{a3}\delta^{b3}\right)^{\pi^+} + \left(-\frac{4}{9}\delta^{ab} + \delta^{a3}\delta^{b3}\right)^{\pi^-} - 2\left(-\frac{4}{9}\delta^{ab} + \delta^{a3}\delta^{b3}\right)^{\pi^0}$$

$$= \left(-\frac{4}{9}\right) + \left(-\frac{4}{9}\right) - 2\left(-\frac{4}{9} + 1\right) = -2. \tag{22}$$

Therefore the final expression for $F_2^{I=2}(x_B)$ has the form

$$F_2^{I=2}(x_B) = g^2 \frac{x_B^2}{Q^2} \frac{F_\pi^2}{2} \left\{ \phi_\pi(x_B) \int_0^1 dz \frac{\phi_\pi(z)}{z(1-z)} \right.$$

$$\left. - \int_0^{1-x_B} dz \phi_\pi(z) \phi_\pi(x_B + z) \left[\frac{1}{z(1-z)} + \frac{1}{(x_B+z)(1-x_B-z)}\right] \right\}. \tag{23}$$

4. Using the large N_c relation (23) one can see that our result obeys the normalization constraint

$$\int_0^1 dx_B \ F_2^{I=2}(x_B) = g^2 \frac{F_\pi^2}{2Q^2} \tag{24}$$

independently of the shape of pion distribution amplitude $\phi_\pi(x)$. We see that the first Mellin moment of $F_2^{I=2}(x_B)$ in the large N_c limit is given solely in terms of the pion decay constant F_π and not sensitive to the details of internal structure of the pion. Recently this moment was computed in the lattice QCD [7] with the results

$$\int_0^1 dx_B \ F_2^{I=2}(x_B) = 0.27(10) \ g^2 \frac{F_\pi^2}{Q^2}. \tag{25}$$

This calculation gives about two times smaller result for the first Mellin moment of $F_2^{I=2}(x_B)$ than our model independent large N_c result. Taking into account various approximations

standing behind our result (24) [leading order of the $1/N_c$ expansion] and the lattice result (25) [quenched approximation, extrapolation in light quark masses etc.] we find that the agreement between the two values is quite reasonable. Note that in contrast to the euclidean lattice calculations which can access the Mellin moments of parton distribution our approach allows us to compute the shape of the twist-4 distributions in mesons.

Let us now compute the third Mellin moment of $F_2^{I=2}(x_B)$. This moment in the large N_c limit is sensitive to the form of the pion distribution amplitude $\phi_\pi(x)$. To demonstrate this we expand the pion distribution amplitude in Gegenbauer series

$$\phi_\pi(x) = 6x(1-x)\left[1 + \sum_{n=2}^{\infty} a_n C_n^{3/2}(2x-1)\right], \tag{26}$$

where $C_n^{3/2}(2x-1)$ are Gegenbauer polynomials. It is a matter of a simple calculation to see that the third Mellin moment of $F_2^{I=2}(x_B)$ is sensitive only to the coefficient a_2 in the Gegenbauer expansion of the pion distribution amplitude (26)

$$\int_0^1 dx_B \; x_B^2 F_2^{I=2}(x_B) = g^2 \frac{F_\pi^2}{Q^2} \frac{3}{10}\left(1 + a_2\right). \tag{27}$$

Generically the Nth Mellin moment (for odd N) can be expressed in terms of Gegenbauer coefficients up to order $N-1$.

Using the general large N_c expression (23) we compute the twist-4 structure function $F_2^{I=2}(x_B)$ for the asymptotic pion distribution amplitude

$$\phi_\pi(x) = 6x(1-x). \tag{28}$$

The result (see Fig. 1) is given by

$$F_2^{I=2}(x_B) = 6\frac{F_\pi^2}{Q^2} g^2 x_B^2 (1 - x_B)(-1 + 2x_B + 2x_B^2). \tag{29}$$

We see that function $F_2^{I=2}(x_B)$ is concentrated at relatively large values of x_B. Let us note that the isospin-2 pion structure function should have at least one zero due to the general sum rule

$$\int_0^1 \frac{dx_B}{x_B^2} \; F_2^{I=2}(x_B) = 0, \tag{30}$$

which directly follows from eq. (2) and holds already at finite N_c.

We have demonstrated that in the large N_c limit the higher twist quark-antiquark correlations in mesons are expressible completely in terms of their twist-2 distribution amplitudes. This model-independent result provides an illustration of general relations between higher twist structure functions and hadron wave functions as recently discussed by S.J. Brodsky [8].

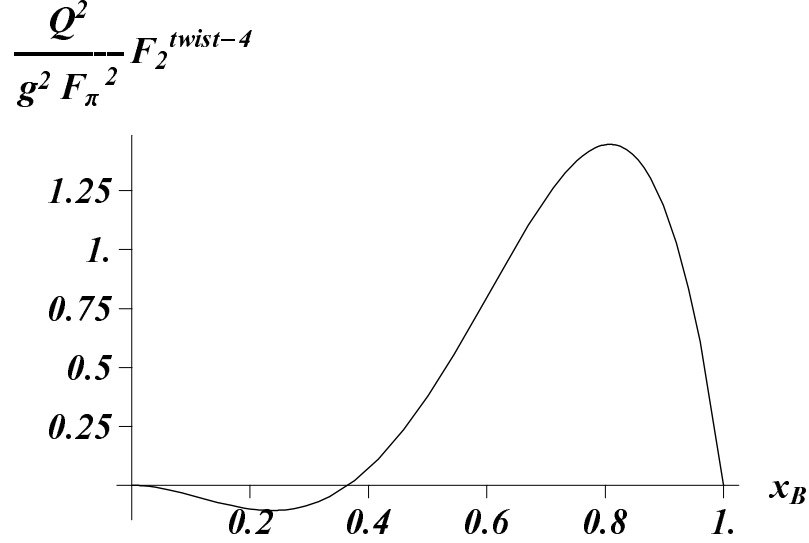$$\frac{Q^2}{g^2 F_\pi{}^2} F_2{}^{twist-4}$$

Figure 1: *The result for $\frac{Q^2}{g^2 F_\pi^2} F_2^{I=2}(x_B)$ in the large N_c limit for the case of the asymptotic pion distribution amplitude.*

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